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Rigorous Derivation of the Cubic NLS in Dimension One

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We derive rigorously the one-dimensional cubic nonlinear Schrödinger equation from a many-body quantum dynamics. The interaction potential is rescaled through a weak-coupling limit together with a short-range one. We start from a factorized initial state, and prove propagation of chaos with the usual two-step procedure: in the former step, convergence of the solution of the BBGKY hierarchy associated to the many-body quantum system to a solution of the BBGKY hierarchy obtained from the cubic NLS by factorization is proven; in the latter, we show the uniqueness for the solution of the infinite BBGKY hierarchy.

KEY WORDS: quantum mechanics, nonlinear schrödinger, gross-pitaevskii, propagation of chaos, BBGKY hierarchy

1. INTRODUCTION

Consider a system of N identical bosons in dimension one, interacting via a pair potential U. According to the axioms of quantum mechanics, the time evolution of the wave function $\Psi_N(t; X_N)$ of the system is ruled by a many-body Schrödinger equation, which in suitable units reads

$$i\partial_t \Psi_N(t; X_N) = -\sum_{j=1}^{N} \partial_{x_j}^2 \Psi_N(t; X_N) + \sum_{1\le j<k\le N} U(x_j - x_k)\Psi_N(t; X_N). \quad (1.1)$$

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where $X_N = (x_1, \ldots, x_N) \in \mathbb{R}^N$ is the collection of the coordinates of the particles. We recall that $\Psi_N(t)$ is an element of $L^2(\mathbb{R}^N)$ and, due to the undistinguishability of particles, satisfies the symmetry property

$$\Psi_N(t; x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = \Psi_N(t; x_1, \ldots x_N). \tag{1.2}$$

for any permutation σ of the indices $(1, \ldots, N)$.

In this note we show that if the number of particles goes to infinity then the system can be described by the non linear one-particle equation

$$i\partial_t \psi(t; x) = -\partial_x^2 \psi(t; x) + \alpha |\psi(t; x)|^2 \psi(t; x), \tag{1.3}$$

provided that the initial data for Eq. (1.1) consist of a factorized state $\Psi_N^I = (\psi^I)^{\otimes N}$, and that the interaction potential is suitably rescaled. More specifically, we require that the strength of the potential decreases like the inverse of the total number of particles, as in the mean-field limit, and simultaneously its range shrinks to zero, as in a short-range limit. The two procedures are combined together in the following way:

$$U(x) = N^{\gamma - 1} V(N^\gamma x) \tag{1.4}$$

with $0 < \gamma < 1$. Then in our model the short range limit is performed more slowly than the mean-field one. The coefficient α appearing in (1.3) equals $\int_{\mathbb{R}} V(x)dx$.

The mean-field (or weak coupling) scaling for quantum systems with infinitely many degrees of freedom was first recognized by Hepp[11] to result in the Hartree equation. Afterwards, Ginibre and Velo[10] extended the analysis to singular potentials.

In the formalism of the first quantization, the convergence was established by Spohn,[14] and twenty years later[3] the problem was split in two different issues: the question of the convergence of the hierarchy of marginals generated by Eq. (1.1) to the one generated by Eq. (1.3), and the question of the uniqueness of the solution of the latter. Following such strategy and developing a new idea for the energy estimate, Erdős and Yau derived the Schrödinger-Poisson equation as the mean-field limit for a system of bosons interacting through a Coulomb potential.[9] The same result was reviewed in,[2] where the proof of the uniqueness for the resulting hierarchy is obtained from a Nirenberg's extension of the Cauchy-Kowalewski theorem.

All the quoted results concern three-dimensional systems and end up in a Hartree-like equation, namely an equation like (1.3) except that the nonlinearity is given by $(V \star |\psi|^2)\psi$ instead of $|\psi|^2\psi$.

On the other hand, the local cubic nonlinearity arises in the Gross-Pitaevskii limit for a gas of bosons in dimension three. The correct scaling was found by Lieb, Seiringer and Yngvason (see Ref. 12 and references therein) in their investigation on the structure of the ground state for a Bose-Einstein condensate.

Afterwards, Erdős, Schlein and Yau[4,6] with Elgart analyzed the same problem in the time-dependent framework. In these two papers the first of the the two-step strategy (i.e. convergence) was accomplished, while the problem of the uniqueness for the solution of the infinite hierarchy was left untouched.

Another partial result towards the cubic local nonlinearity was obtained in (Ref. 1) for a gas of one dimensional bosons interacting through a Dirac's delta potential, and still the problem of the uniqueness of the solution of the infinite hierarchy was not solved.

Let us stress that, in spite of its simplicity, the one-dimensional case is physically meaningful, since Eq. (1.3) is used to describe boson gas in elongated traps and the so-called cigar-shaped Bose-Einstein condensates.[13] Of course, the three-dimensional problem has a more general reach and is more difficult from the technical point of view. Furthermore, we stress that the scaling we are considering here is not the one-dimensional analogue of the one found by Lieb, Seiringer and Yngvason, that requires $\gamma = 1$ and results in a nonlinearity which is still cubic but whose strength is given by the scattering length of the unscaled potential.

During the final draft of this paper we were made aware of the fact that Erdős, Schlein and Yau had achieved the proof of the uniqueness for the infinite hierarchy in the three-dimensional setting,[5] and in the time between the submission of our paper and the draft of the revised version, they completed the proof for the three-dimensional case in the scaling of Lieb, Seiringer and Yngvason.[7,8]

Before stating the result, let us recall some standard definitions.
We make use of the shorthand notation

$$\begin{aligned} X_j &:= (x_1, \ldots, x_j) \in \mathbb{R}^j \\ X_k^j &:= (x_j, \ldots, x_k) \in \mathbb{R}^{k-j+1} \end{aligned} \tag{1.5}$$

A state of the system is represented by a square integrable function $\Psi_N(t)$ satisfying the normalization condition

$$\int_{\mathbb{R}} |\Psi_N(t; X_N)|^2 \, dX_N = 1 \tag{1.6}$$

The same state can be equivalently denoted by the orthogonal projection $\rho_N(t)$ on the linear span of $\Psi_N(t)$ as a subspace $L^2(\mathbb{R}^N)$. The integral kernel of $\rho_N(t)$ reads

$$\rho_N(t; X_N; Y_N) = \Psi_N(t; X_N)\overline{\Psi_N(t; Y_N)} \tag{1.7}$$

and its action on any $\Phi \in L^2(\mathbb{R}^N)$ is given by

$$(\rho_N(t)\Phi)(X_N) = \int_{\mathbb{R}^N} \rho_N(t; X_N; Y_N)\Phi(Y_N)\, dY_N \tag{1.8}$$

Any set of n particles among the N in the system is described by the reduced density matrix (sometimes called correlation function or marginal), namely the integral operator defined by the kernel

$$\rho_{N,n}(t; X_n; Y_n) = \int_{\mathbb{R}^{N-n}} \rho_N\big(t; X_n, Z_N^{n+1}; Y_n, Z_N^{n+1}\big)\, dZ_N^{n+1}. \tag{1.9}$$

Due to (1.6), $\rho_{N,n}(t)$ is a positive operator whose trace equals one, i.e.

$$\|\rho_{N,n}(t)\|_{\mathcal{L}^1(L^2(\mathbb{R}^n))} = 1 \tag{1.10}$$

where we denoted by $\mathcal{L}^1(L^2(\mathbb{R}^n))$ the space of trace-class operators on $L^2(\mathbb{R}^n)$. Analogously, we denote by $\mathcal{L}^2(L^2(\mathbb{R}^n))$ the space of the Hibert-Schmidt operators on $L^2(\mathbb{R}^n)$. It is well known that $\mathcal{L}^1(L^2(\mathbb{R}^n)) \subset \mathcal{L}^2(L^2(\mathbb{R}^n))$ and

$$\|\rho(t)\|_{\mathcal{L}^2(L^2(\mathbb{R}^n))} \leq \|\rho(t)\|_{\mathcal{L}^1(L^2(\mathbb{R}^n))} \tag{1.11}$$

for any $\rho(t) \in \mathcal{L}^1(L^2(\mathbb{R}^n))$.

As described in[9], one can define the so-called Sobolev spaces of density matrices, denoted by $\mathcal{L}^{m,p}(L^2(\mathbb{R}^n))$. Let S_j be the operator $(\mathbb{I} - \partial_j^2)^{\frac{1}{2}}$. Then, $\rho \in \mathcal{L}^{m,p}(L^2(\mathbb{R}^n))$ if

$$\mathrm{Trace}\big(\big|S_1^m \ldots S_n^m \rho(t) S_1^m \ldots S_n^m\big|^p\big) < \infty \tag{1.12}$$

and

$$\|\rho\|_{\mathcal{L}^{m,p}(L^2(\mathbb{R}^n))} = \big[\mathrm{Trace}\big(\big|S_1^m \ldots S_n^m \rho S_1^m \ldots S_n^m\big|^p\big)\big]^{\frac{1}{p}}. \tag{1.13}$$

In the following we mainly use the space $\mathcal{L}^{1,2}(L^2(\mathbb{R}^n))$. We recall here that the norm in such space has a simple expression in terms of the integral kernel of $\rho(t)$, namely

$$\|\rho(t)\|^2_{\mathcal{L}^{1,2}(L^2(\mathbb{R}^n))} = \int_{\mathbb{R}^{2n}} \left| \left[\prod_{j=1}^n \big(1 - \partial_{x_j}^2\big)^{1/2} \big(1 - \partial_{y_j}^2\big)^{1/2} \right] \rho(t; X_n, Y_n) \right|^2 dX_n\, dY_n. \tag{1.14}$$

For the sake of simplicity we use the notation

$$E_n := \mathcal{L}^{1,2}(L^2(\mathbb{R}^n)). \tag{1.15}$$

Introducing the Fourier representation

$$\rho(t; X_n; Y_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{iX_n \cdot \Xi_n + iY_n \cdot \Lambda_n} \hat{\rho}_{N,n}(t; \Xi_n; \Lambda_n)\, d\Xi_n\, d\Lambda_n \tag{1.16}$$

with variables in the Fourier space denoted as follows

$$\Xi_n = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \quad \Lambda_n = (\lambda_1, \ldots \lambda_n) \in \mathbb{R}^n \tag{1.17}$$

we can express the norm in E_n in (1.14) as

$$\|\rho(t)\|^2_{E_n} = \int_{\mathbb{R}^{2n}} \Gamma(\Xi_n)^2 \Gamma(\Lambda_n)^2\, |\hat{\rho}(t; \Xi_n; \Lambda_n)|^2\, d\Xi_n\, d\Lambda_n \tag{1.18}$$

where we defined the function

$$\Gamma(W) = \prod_{j=1}^{k} \left(1 + w_j^2\right)^{\frac{1}{2}} \tag{1.19}$$

for any $W = (w_1, \ldots, w_k) \in \mathbb{R}^k$.

Finally, we recall that, for the problem defined by (1.1) and (1.4), the integral kernels of the reduced density matrices $\rho_{N,n}(t)$ solve the finite BBGKY hierarchy, i.e.

$$\begin{aligned} i\partial_t \rho_{N,n}(t; X_N; Y_N) = &-\sum_{j=1}^{n} \left(\partial_{x_j}^2 - \partial_{y_j}^2\right) \rho_{N,n}(t; X_n; Y_n) \\ &+ N^{\gamma-1} \sum_{1 \le j < k \le n} [V(N^\gamma(x_j - x_k)) \\ &- V(N^\gamma(y_j - y_k))] \rho_{N,n}(t; X_n; Y_n) \\ &+ \frac{N-n}{N^{1-\gamma}} \sum_{j=1}^{n} \int_{\mathbb{R}} [V(N^\gamma(x_j - z)) \\ &- V(N^\gamma(y_j - z))] \rho_{N,n+1}(t; X_n, z; Y_n, z)\, dz. \end{aligned} \tag{1.20}$$

We write here for completeness the so-called infinite BBGKY hierarchy

$$\begin{aligned} i\partial_t \rho_n(t; X_n; Y_n) = &-\sum_{j=1}^{n} \left(\partial_{x_j}^2 - \partial_{y_j}^2\right) \rho_n(t; X_n; Y_n) \\ &+ \alpha \sum_{1 \le i \le n} [\rho_{n+1}(t; X_n, x_i; Y_n, x_i) - \rho_{n+1}(t; X_n, y_i; Y_n, y_i)]. \end{aligned} \tag{1.21}$$

where, as already mentioned,

$$\alpha = \int_{\mathbb{R}} V(x)dx. \tag{1.22}$$

Our main result is the following.

Theorem 1.1. *Consider the Cauchy problem (1.1), where the potential U is given in (1.4) with V a non negative function in the Schwarz space $\mathcal{S}(\mathbb{R})$, and factorized initial data*

$$\Psi_N^I := (\psi^I)^{\otimes N}. \tag{1.23}$$

Denoted by ρ_N^I the orthogonal projection in $L^2(\mathbb{R}^n)$ on the linear span of Ψ_N^I, and by H_N the N-particle Hamilton operator

$$H_N := -\sum_{j=1}^{N} \partial_{x_j}^2 + N^{\gamma-1} \sum_{1\le j<k\le N} V(N^\gamma(x_j - x_k)) \tag{1.24}$$

with $0 < \gamma < 1$, we assume that for any n there exists $N^\star(n)$ s.t. if $N > N^\star(n)$ then the following growth condition holds

$$\left(\Psi_N^I, H_N^n \Psi_N^I\right) \le M^n N^n \tag{1.25}$$

for some $M > 0$.

Then, the n-particle reduced density matrix satisfies

$$\rho_{N,n} \longrightarrow (\psi \otimes \bar{\psi})^{\otimes n}, \quad N \longrightarrow \infty \tag{1.26}$$

in the weak-$\star$ topology of the space $L^\infty(\mathbb{R}, E_n)$, where E_n has been defined in (1.15).

The function ψ appearing in (1.26) satisfies Eq. (1.3) with initial data ψ^I, and

$$\alpha := \int_{\mathbb{R}} V(x)\, dx. \tag{1.27}$$

The paper is organized as follows. In Sec. 2 we derive the two estimates needed to prove the result. In Sec. 3 we prove the convergence, up to subsequences, of any solution of (1.20) to a solution of (1.21). In Sec. 4 we prove uniqueness for solutions of the infinite hierarchy and identify such solution with the marginals generated by the solutions of Eq. (1.3). Finally, in Sec. 5 we prove the existence of good factorized initial data, under the restrictive hypothesis $\gamma < 1/2$.

2. ESTIMATES

The first estimate is the one dimensional version of inequality (3.25) in Ref. 4.

Proposition 2.1. *Let V be a non negative function belonging to the Schwarz space $\mathcal{S}(\mathbb{R})$, H_N be defined like in (1.24), and ρ_N^I be the initial density matrix of the system, and assume that it satisfies the inequality (1.25) for some $M \in \mathbb{R}$ and, eventually in N, for any $n \le N$. Then, for any $M_1 > M$ and any $n \in \mathbb{N}$ there exists $\bar{N}$ depending on n and M_1 such that*

$$\|\rho_{N,n}(t)\|_{\mathcal{L}^{1,1}(L^2(\mathbb{R}^n))} := \text{Trace}\left[\left(\prod_{j=1}^{n} S_j\right)\rho_{N,n}(t)\left(\prod_{j=1}^{n} S_j\right)\right] \le M_1^n \tag{2.1}$$

for any time t and $N \geq \max(\bar{N}, N^{\star}(n))$, *where* $N^{\star}(n)$ *has been defined in Theorem 1.1.*

Proof. We refer the reader to the proof of Proposition 3.1. and Corollary 3.2. in Ref. 4. Let us just explain why in this case we can prove estimate (2.1) for $\gamma < 1$ while in Ref. 4 the best one can get is $\gamma < 3/5$. The one dimensional setting differs from the three dimensional one by the fact that the following inequality holds

$$\langle V'_{12} \rangle \leq \frac{\| V' \|_{L^1(\mathbb{R})}}{2a} \langle S_1^2 \rangle \tag{2.2}$$

where V_{ij} is the multiplication by $a^{-1} V(a^{-1}(x_i - x_j))$, and $\langle A \rangle$ denotes the mean value of the observable A on some function belonging to the domain of S_1^2. Therefore, inequality (3.23) in Ref. 4 is replaced by

$$2\mathrm{Re} \left\langle V_{12} \prod_{j=1}^{n+1} S_j^2 \right\rangle \geq -\frac{\| V' \|_{L^1(\mathbb{R})}}{2a} (2\alpha_1 + 2\alpha_1^{-1} + \alpha_2 + \alpha_3) \left\langle \prod_{j=1}^{n+1} S_j^2 \right\rangle - \left(\frac{\| V' \|_{L^1(\mathbb{R})}}{2a\alpha_2} + \frac{\| V' \|_{L^\infty(\mathbb{R})}}{a^2 \alpha_3} \right) \left\langle S_1^4 \prod_{j=2}^{n+1} S_j^2 \right\rangle. \tag{2.3}$$

Analogously

$$2\mathrm{Re} \left\langle V_{1,n+2} \prod_{j=1}^{n+1} S_j^2 \right\rangle \geq -\frac{\| V' \|_{L^1(\mathbb{R})}}{2a} (\alpha_4 + \alpha_4^{-1}) \left\langle \prod_{j=1}^{n+2} S_j^2 \right\rangle \tag{2.4}$$

where all the positive coefficients a, α_j's can be chosen arbitrarily. Recalling that $a = N^{-\gamma}$, and setting $\alpha_1 = \alpha_2 = \alpha_4 = 1$, and $\alpha_3 = N^{\gamma}$, the following inequality is proven (see formulas (3.22) and (3.24) in Ref. 4)

$$\left\langle (H_N + N) \left(\prod_{j=1}^{n+2} S_j^2 \right) (H_N + N) \right\rangle \geq N^2 f(N, n) \left\langle \prod_{j=1}^{n+2} S_j^2 \right\rangle + N g(N, n) \left\langle S_1^4 \prod_{j=2}^{n+1} S_j^2 \right\rangle \tag{2.5}$$

where

$$f(N, n) := \left(1 - \frac{n}{N}\right) \left[\left(1 - \frac{n+1}{N}\right) - \frac{\| V' \|_{L^1(\mathbb{R})}}{4} n(n+1)(5N^{\gamma-2} + N^{2\gamma-2}) \right]$$

$$g(N, n) := \left(1 - \frac{n}{N}\right) \left[(2n+1) - (\| V' \|_{L^1(\mathbb{R})} + 2\| V' \|_{L^\infty(\mathbb{R})}) \frac{n+1}{4} N^{\gamma-1} \right] \tag{2.6}$$

Since for $N \to \infty$ $f(N,n) \to 1$ and $g(N,n) \to 2n+1$, then for any positive $C < 1$ there exists $\bar{N}$ such that $N > \bar{N}$ implies

$$\left\langle (H_N + N)\left(\prod_{j=1}^{n+2} S_j^2\right)(H_N + N)\right\rangle \geq C^2 N^2 \left\langle \prod_{j=1}^{n+2} S_j^2 \right\rangle \tag{2.7}$$

From inequality (2.7) one can prove by induction on n that, for any positive $C < 1$, there exists $\tilde{N}$ depending on C and on n such that if $N > \tilde{N}$ then

$$\langle (H_N + N)^n \rangle \geq C^n N^n \left\langle \prod_{j=1}^{n} S_j^2 \right\rangle \tag{2.8}$$

for any t. Following Corollary 3.2. in Ref. 4, one applies the conservation law of $\langle (H_N + N)^n \rangle$ to estimate (2.8) and the proof is complete. □

Remark 2.2. Since the space of trace class operators is embedded in the space of the Hilbert-Schmidt operators one obviously has

$$\|\rho_{N,n}(t)\|_{E_n} \leq \text{Trace}\left[\left(\prod_{j=1}^{n} S_j\right)\rho_{N,n}(t)\left(\prod_{j=1}^{n} S_j\right)\right] \leq M_1^n \tag{2.9}$$

for any time t and $N \geq \max(\bar{N}, N^\star(n))$.

Remark 2.3. For a factorized initial state, the condition (1.25) implies that ψ^I is a function in the Schwarz class. A costruction of a suitable ψ^I is made in the appendix, under the additional hypothesis $\gamma < 1/2$.

The second estimate we need concerns the norm of density matrices in the space E_n.

Proposition 2.4. *Given a density matrix $\rho \in E_{n+1}$, let $\rho(X_{n+1}; Y_{n+1})$ be its integral kernel and consider the function $\sigma : \mathbb{R}^{2n} \to \mathbb{C}$ defined by*

$$\sigma(X_n; Y_n) = \int_{\mathbb{R}} U(x_1 - z)\,\rho(X_n, z; Y_n, z)\,dz. \tag{2.10}$$

where U has a bounded Fourier transform $\hat{U}$. Then, the integral operator σ having $\sigma(X_n; Y_n)$ as integral kernel belongs to E_n and

$$\|\sigma\|_{E_n} \leq \sqrt{32\pi}\,\|\hat{U}\|_{L^\infty(\mathbb{R})}\|\rho\|_{E_{n+1}} \tag{2.11}$$

Proof. Let us define the function

$$\Theta : \mathbb{R}^{2n+1} \to \mathbb{C},\ \ \Theta(X_n; Y_n; z) := \rho(X_n, z; Y_n, z). \tag{2.12}$$

By elementary computation one gets

$$\hat{\sigma}(\Xi_n;\Lambda_n) = \int_{\mathbb{R}} \hat{U}(k)\hat{\Theta}\big(\xi_1 - k,\, \Xi_n^2;\Lambda_n;k\big)\,dk \tag{2.13}$$

where, according to notation (1.5), we wrote $\Xi_n^2 := (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1}$, and then

$$\begin{aligned}\|\sigma\|_{E_n}^2 &:= \int_{\mathbb{R}^{2n}} \Gamma(\Xi_n)^2\Gamma(\Lambda_n)^2\,|\hat{\sigma}(\Xi_n;\Lambda_n)|^2\;d\Xi_n\,d\Lambda_n\\ &= \int_{\mathbb{R}^{2n}} \Gamma\big(\Xi_n^2\big)^2\Gamma(\Lambda_n)^2\left|\int_{\mathbb{R}} \Gamma(\xi_1)\hat{U}(k)\hat{\Theta}\big(\xi_1 - k,\, \Xi_n^2;\Lambda_n;k\big)\,dk\right|^2 d\Xi_n\,d\Lambda_n.\end{aligned} \tag{2.14}$$

where the function Γ has been defined in (1.19). Since

$$\Gamma(\xi_1) \leq \sqrt{2}[\Gamma(\xi_1 - k) + \Gamma(k)] \tag{2.15}$$

we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \Gamma(\Xi_n)^2\Gamma(\Lambda_n)^2\,|\hat{\sigma}(\Xi_n;\Lambda_n)|^2\;d\Xi_n\,d\Lambda_n\\ &\leq 4\int_{\mathbb{R}^{2n}} \Gamma\big(\Xi_n^2\big)^2\Gamma(\Lambda_n)^2\left|\int_{\mathbb{R}} \Gamma(\xi_1 - k)\hat{U}(k)\hat{\Theta}(\xi_1 - k,\, \Xi_n^2;\Lambda_n;k)\,dk\right|^2 d\Xi_n\,d\Lambda_n\\ &+4\int_{\mathbb{R}^{2n}} \Gamma\big(\Xi_n^2\big)^2\Gamma(\Lambda_n)^2\left|\int_{\mathbb{R}} \Gamma(k)\hat{U}(k)\hat{\Theta}(\xi_1 - k,\, \Xi_n^2;\Lambda_n;k)\,dk\right|^2 d\Xi_n\,d\Lambda_n\\ &\leq 4\int_{\mathbb{R}^{2n}} \Gamma\big(\Xi_n^2\big)^2\Gamma(\Lambda_n)^2\left|\int_{\mathbb{R}} \Gamma(\xi_1 - k)\Gamma(k)\frac{\hat{U}(k)}{\Gamma(k)}\hat{\Theta}(\xi_1 - k,\, \Xi_n^2;\Lambda_n;k)dk\right|^2 d\Xi_n\,d\Lambda_n\\ &+4\int_{\mathbb{R}^{2n}} \Gamma\big(\Xi_n^2\big)^2\Gamma(\Lambda_n)^2\left|\int_{\mathbb{R}} \Gamma(\xi_1 - k)\Gamma(k)\frac{\hat{U}(k)}{\Gamma(\xi_1 - k)}\hat{\Theta}(\xi_1 - k,\, \Xi_n^2;\Lambda_n;k)dk\right|^2 d\Xi_n d\Lambda_n.\end{aligned} \tag{2.16}$$

Applying the Cauchy-Schwarz inequality to both integrals we get

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \Gamma(\Xi_n)^2\Gamma(\Lambda_n)^2\,|\hat{\sigma}(\Xi_n;\Lambda_n)|^2\;d\Xi_n\,d\Lambda_n\\ &\leq 8\pi\int_{\mathbb{R}^{2n}} \Gamma\big(\Xi_n^2\big)^2\Gamma(\Lambda_n)^2\int_{\mathbb{R}} \Gamma(\xi_1 - k)^2\Gamma(k)|\hat{U}(k)\hat{\Theta}(\xi_1 - k,\, \Xi_n^2;\Lambda_n;k)|^2\,dk\,d\Xi_n\,d\Lambda_n\\ &\leq 8\pi\,\|\hat{U}\|_\infty^2\int_{\mathbb{R}^{2n+1}} \Gamma(\zeta)^2\Gamma(\Xi_n)^2\Gamma(\Lambda_n)^2|\Theta(\Xi_n;\Lambda_n;\zeta)|^2 d\Xi_n d\Lambda_n d\zeta\end{aligned} \tag{2.17}$$

We observe that

$$\Theta(X_n; Y_n; x_{n+1}) = \int_{\mathbb{R}} \delta(x_{n+1} - y_{n+1})\, \rho(X_{n+1}; Y_{n+1})\, dy_{n+1}$$

where δ is the Dirac's measure and has a bounded Fourier transform $\hat{\delta} = (2\pi)^{-1/2}$. Therefore,

$$\int_{\mathbb{R}^{2n+1}} \Gamma(\zeta)^2 \Gamma(\Xi_n)^2 \Gamma(\Lambda_n)^2 \left| \hat{\Theta}(\Xi_n; \Lambda_n; \zeta) \right|^2 d\Xi_n\, d\Lambda_n\, d\zeta < \infty \tag{2.18}$$

and applying estimate (2.17) we conclude

$$\|\sigma\|_{E_n}^2 \leq 32\pi \|\hat{U}\|_{L^\infty(\mathbb{R})}^2 \|\rho\|_{E_{n+1}}^2 \tag{2.19}$$

□

Remark 2.5. If $U(x) = N^{\gamma-1} V(N^\gamma x)$, then

$$\|\sigma\|_{E_n}^2 \leq \frac{32\pi}{N^2} \|\hat{V}\|_{L^\infty(\mathbb{R})}^2 \|\rho\|_{E_{n+1}}^2. \tag{2.20}$$

Furthermore, if V is integrable, then

$$\|\sigma\|_{E_n}^2 \leq \frac{16}{N^2} \|V\|_{L^1(\mathbb{R})}^2 \|\rho\|_{E_{n+1}}^2. \tag{2.21}$$

3. CONVERGENCE TO THE INFINITE BBGKY HIERARCHY

Following the strategy of Ref. 3 we prove convergence in some weak sense of the solutions to the finite BBGKY hierarchy to solutions to the infinite one. The first point is to construct a converging subsequence. We have in fact the following proposition.

Proposition 3.1. *There exists an increasing function $\phi : \mathbb{N}^* \to \mathbb{N}^*$ such that for any $n \in \mathbb{N}^*$, the sequence $\{\rho_{\phi(N),n}\}_{N \in \mathbb{N}^*}$ converges in the space $L^\infty(\mathbb{R}, E_n)$ in the sense of the weak-$\star$ topology.*

The proof follows the one of Proposition 4.1. In Ref. 1, except for the irrelevant fact that in our case the norm of $\rho_{N,n}$ is uniformly bounded with respect to N only.

Remark 3.2. Such a convergence result holds in the space $\prod_{n=1}^\infty L^\infty(\mathbb{R}, E_n)$ endowed with the product topology. Due to estimate (2.9) on can establish convergence in the sphere of the space $L^\infty(\mathbb{R}, E_n)$ with radius M_1^n and centered at zero. Therefore, for any limit point $\rho_n(t)$, inequality (2.9) gives

$$\|\rho_n(t)\|_{E_n} \leq M_1^n. \tag{3.1}$$

In the following theorem we prove the weak convergence of the finite BBGKY hierarchy to the infinite one.

Theorem 3.3. *Consider the Cauchy problem (1.1) where the potential U is given in (1.4), $V \in \mathcal{S}(\mathbb{R})$, $V \geq 0$, and the initial data $\Psi_N^I \in H^1(\mathbb{R}^N)$ satisfy the undistinguishability assumption (1.2) and the normalization condition (1.6). Moreover, denoted by ρ_N^I the orthogonal projection in $L^2(\mathbb{R}^N)$ on the linear span of Ψ_N^I, we assume that the growth condition (1.25) is satisfied.*

Let us assume that the n-particle marginal $\rho_{N,n}^I$, converges to an operator ρ_n^I for $N \to \infty$ in the weak-⋆ topology of the space E_n defined in (1.15).

Let $\Psi_N \in C^0(\mathbb{R}; H^1(\mathbb{R}^N))$ be the mild solution of the Cauchy problem (1.1), (1.4), with initial data Ψ_N^I, $\rho_N(t)$ the orthogonal projection in $L^2(\mathbb{R}^n)$ on the space spanned by $\Psi_N(t)$, and $\rho_{N,n}(t)$ its n-particle marginal.

Then, any simultaneous limit point as $N \to \infty$ of the family of partial traces $\rho_{N,n}$ solves the infinite Schrödinger hierarchy (1.21) in the sense of distributions $\mathcal{D}'(\mathbb{R}^{2n+1})$.

Here, limit points are understood in the sense of the product topology on $\prod_{n=1}^{\infty} L^\infty(\mathbb{R}; E_n)$, each factor being equipped with the weak-⋆ topology.

Proof. Let us consider a converging subsequence $\{\rho_{N_j,n}\}_{j\in\mathbb{N}}$ of marginals. Hereafter we omit the indication of the variable j and simply write $\rho_{N,n}$. Let ρ_n denote the limit of $\rho_{N,n}$ as N goes to infinity, in the weak-⋆ topology of $L^\infty(\mathbb{R}; E_n)$.

We show that the integral kernel $\rho_n(t; X_n; Y_n)$ of the operator ρ_n satisfies the infinite BBGKY hierarchy (1.21) in the sense of distributions $\mathcal{D}'(\mathbb{R}^{2n+1})$.

It is easily seen that the weak-⋆ convergence in $L^\infty(\mathbb{R}, E_n)$ implies the weak-⋆ convergence in $L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$ and consequently the convergence in $\mathcal{D}'(\mathbb{R}^{2n+1})$ for the function $\rho(t; X_n; Y_n)$. It follows that the l.h.s. and the laplacian term of (1.20) converge to the corrisponding terms of (1.21).

We observe that $\rho_{N,n}(t) \in L^\infty(\mathbb{R}^{2n})$. Indeed, by the ordinary inverse Fourier transform formula (1.16) and using the Cauchy-Schwarz inequality one has

$$\|\rho_{N,n}(t)\|_{L^\infty(\mathbb{R}^{2n})} \leq \left(\frac{\pi}{2}\right)^n \|\rho_{N,n}(t)\|_{E_n}. \tag{3.2}$$

Consider the second term in the r.h.s. of (1.20). It consists of a sum of $\frac{n(n-1)}{2}$ terms whose generic term, evaluated on a test function $\varphi \in \mathcal{D}(\mathbb{R}^{2n+1})$, gives

$$\begin{aligned} N^{\gamma-1} \int_{\mathbb{R}^{2n+1}} & V(N^\gamma(x_1 - x_2))\rho_{N,n}(t; X_n; Y_n)\varphi(t; X_n; Y_n)\, dt\, dX_n\, dY_n \\ & \leq N^{\gamma-1} \left(\frac{\pi}{2}\right)^n |\mathrm{Supp}(\varphi)| \|V\|_{L^\infty(\mathbb{R})} \|\rho_{N,n}\|_{L^\infty(\mathbb{R},E_n)} \|\varphi\|_{L^\infty(\mathbb{R}^{2n+1})} \end{aligned} \tag{3.3}$$

where $|\mathrm{Supp}(\varphi)|$ denotes the Lebesgue measure of the support of φ. It appears from (3.3) that the first sum in the r.h.s of (1.20) vanishes as N goes to infinity in the sense of distributions.

It remains to prove that the last sum in the r.h.s. of (1.20) converges to the corresponding term in the r.h.s. of (1.21). It suffices to prove the convergence for one term of the sum, e.g.

$$\int_{\mathbb{R}^{n+1}} \varphi(t;X_n;Y_n)\left[\int_{\mathbb{R}} dz\, N^{\gamma} V(N^{\gamma}(x_1-z))\rho_{N,n+1}(t;X_n,z;Y_n,z)\,dz\right] dt\,dX_n\,dY_n$$
$$\longrightarrow \int_{\mathbb{R}^{n+1}} \varphi(t;X_n;Y_n)\rho_{n+1}(t;X_n,x_1;Y_n,x_1) \tag{3.4}$$

First we prove that the r.h.s. in (3.4) can be written as follows

$$\int_{\mathbb{R}^{n+1}} \varphi(t;X_n;Y_n)\left[\int_{\mathbb{R}} dz\, N^{\gamma} V(N^{\gamma}(x_1-z))\,\rho_{N,n+1}(t;X_n,z;Y_n,z)\,dz\right] dt\,dX_n\,dY_n$$
$$= \langle \Phi_{N,n+1}, \rho_{N,n+1}\rangle_{L^1(\mathbb{R},E^*_{n+1})} \tag{3.5}$$

where $\Phi_{N,n+1} \in L^1(\mathbb{R}, E^*_{n+1})$ and $\langle\cdot,\cdot\rangle_X$ denotes the duality product in the space X. Let us write

$$\langle \Phi_{N,n+1}, \rho_{N,n+1}\rangle_{L^1(\mathbb{R},E^*_{n+1})} = \int_{\mathbb{R}} \langle \Phi_{N,n+1}(t), \rho_{N,n+1}(t)\rangle_{E^*_{n+1}}\,dt \tag{3.6}$$

Our task is then to prove that $\Phi_{N,n+1}(t) \in E^*_{n+1} = \mathcal{L}^{-1,2}(L^2(\mathbb{R}^{n+1}))$. By a standard computation in the Fourier space one has

$$\langle \Phi_{N,n+1}(t), \rho_{N,n+1}(t)\rangle_{E^*_{n+1}}$$
$$= \frac{1}{(2\pi)^{n+\frac{1}{2}}}\int_{\mathbb{R}^{2n+2}} \hat{\varphi}\big(t;-\xi_1-\xi_{n+1}-\lambda_{n+1},-\Xi^2_n;-\Lambda_n\big)$$
$$\hat{V}\left(\frac{\lambda_{n+1}-\xi_{n+1}}{N^{\gamma}}\right)\hat{\rho}_{N,n+1}(t;\Xi_{n+1};\Lambda_{n+1})\,d\Xi_{n+1}\,d\Lambda_{n+1} \tag{3.7}$$

Notice that the function

$$\hat{\Phi}_{N,n+1}(t;\Xi_{n+1};\Lambda_{n+1}) = \frac{1}{(2\pi)^{n+\frac{1}{2}}}\hat{\varphi}(t;-\xi_1-\xi_{n+1}-\lambda_{n+1},-\Xi^2_n;-\Lambda_n)$$
$$\times \hat{V}\left(\frac{\lambda_{n+1}-\xi_{n+1}}{N^{\gamma}}\right) \tag{3.8}$$

satisfies the inequality

$$\int_{\mathbb{R}^{2n+2}} \frac{\left|\hat{\Phi}_{N,n+1}(t;\Xi_{n+1};\Lambda_{n+1})\right|^2}{\Gamma(\Xi_{n+1})^2\Gamma(\Lambda_{n+1})^2}\, d\Xi_{n+1}\, d\Lambda_{n+1} \leq 2\pi \|\hat{\varphi}(t)\|^2_{L^\infty(\mathbb{R}^{2n})}\|\hat{V}\|^2_{L^\infty(\mathbb{R})} < \infty \tag{3.9}$$

Therefore $\Phi_{N,n+1}(t)$ belongs to E^*_{n+1}.
Moreover

$$\begin{aligned}\|\Phi_{N,n+1}\|_{L^1(\mathbb{R},E^*_{n+1})} &= \int_{\mathbb{R}} \|\Phi_{N,n+1}(t)\|_{E^*_{n+1}}\, dt \\ &\leq \sqrt{2\pi}\,\mathrm{Diam}(\mathrm{Supp}[\varphi])\|\hat{\varphi}\|_{L^\infty(\mathbb{R}^{2n+1})}\|\hat{V}\|_{L^\infty(\mathbb{R})} \end{aligned} \tag{3.10}$$

where Diam(Supp[φ]) is the diameter in $\mathbb{R}^{2n+1}$ of the support of φ. We have then proven that the l.h.s. of (3.4) can be interpreted as a duality product in $L^1(\mathbb{R}, E^*_{n+1})$. Furthermore, it appears that, at any t, $\Phi_{N,n+1}(t)$ converges strongly in E^*_{n+1} to the functional $\Phi_{n+1}(t)$ represented by the function $\Phi_{n+1}(t; X_{n+1}; Y_{n+1})$ whose Fourier transform reads

$$\hat{\Phi}_{n+1}(t;\Xi_{n+1};\Lambda_{n+1}) = \frac{1}{(2\pi)^{n+\frac{1}{2}}}\hat{\varphi}(t; -\xi_1 - \xi_{n+1} - \lambda_{n+1}, -\Xi^2_n; -\Lambda_n)\hat{V}(0) \tag{3.11}$$

Therefore, recalling that $\rho_{N,n+1}$ converges weakly-$\star$ to ρ_{n+1}, we have that

$$\begin{aligned}&\langle\Phi_{N,n+1}, \rho_{N,n+1}\rangle_{L^1(\mathbb{R},E^*_{n+1})} \longrightarrow \langle\Phi_{n+1}, \rho_{n+1}\rangle_{L^1(\mathbb{R},E^*_{n+1})} \\ &= \frac{\hat{V}(0)}{(2\pi)^{n+\frac{1}{2}}}\int_{\mathbb{R}^{2n+2}} \hat{\varphi}\big(t; -\xi_1 - \xi_{n+1} - \lambda_{n+1}, -\Xi^2_n; -\Lambda_n\big) \\ &\quad \times \hat{\rho}_{N,n+1}(t;\Xi_{n+1};\Lambda_{n+1})\, d\Xi_{n+1}\, d\Lambda_{n+1} \\ &= \left(\int_{\mathbb{R}} V(x)\, dx\right)\int_{\mathbb{R}^{2n}} \varphi(t; X_n; Y_n)\rho_{n+1}(t; X_n, x_1; Y_n, x_1)\, dX_n, dY_n\end{aligned} \tag{3.12}$$

The argument can be repeated for each term of the last sum in the r.h.s. of (1.20), and then the result is proven. □

4. UNIQUENESS AND SERIES REPRESENTATION

The proof of the uniqueness for the solution to the infinite BBGKY hierarchy (1.21) is easily obtained using estimates (2.1) and (3.1). Due to the linearity of the hierarchy it is sufficient to prove the following result.

Theorem 4.1. *Let* $\{\theta_n(t)\}_{n\geq 1}$ *be a solution of (1.21) in some time interval* $[0, T)$, $T > 0$, *such that* $\theta_n(t) \in E_n$ *for any* $n \geq 1$ *and* $t \in [0, T)$, $\theta_n(0) = 0$, *and* $\|\theta_n(t)\| \leq K^n$ *for some positive* K. *Then,* $\theta_n(t) = 0$.

Proof. We consider the free propagator for density matrices $U_n(t)$, acting on integral kernels as follows

$$[U_n(t)\theta_n](X_n;Y_n) = \frac{1}{(4\pi t)^n}\int_{\mathbb{R}^{2n}} e^{i\frac{|X_n-X_n'|^2-|Y_n-Y_n'|^2}{4t}}\theta_n(X_n';Y_n')\,dX_n'\,dY_n' \tag{4.1}$$

Let us exploit the "interaction representation," namely define

$$u_n(t) := U_n(-t)\theta_n(t) \tag{4.2}$$

Then, $u_n(t)$ solves

$$i\partial_t u_n(t) = U_n(-t)L_{n,n+1}U_{n+1}(t)u_{n+1}(t) \tag{4.3}$$

where the action of the operator $L_{n,n+1}$ is defined by

$$L_{n,n+1}v_{n+1}(t) = \alpha\sum_{i=1}^{n}[v_{n+1}(t;X_n,x_i;Y_n,x_i) - v_{n+1}(t;X_n,y_i;Y_n,y_i)] \tag{4.4}$$

Due to estimate (2.11) with $U=\delta$, we obtain that $L_{n,n+1}$ is a bounded operator mapping E_{n+1} to E_n and

$$\|L_{n,n+1}\|_{\mathcal{L}(E_{n+1},E_n)} \le 8\alpha n \tag{4.5}$$

Furthermore, due to unitarity of U_n in E_n, the growth condition assumed for $\theta_n(t)$ remains valid also for u_n. Applying Theorem (3.1) in Ref. 2 we conclude the proof. □

Corollary 4.2. *The weak-⋆ limit ρ_n of the sequence $\rho_{N,n}$ of reduced density matrix associated to problem (1.1) provided with the scaling (1.4) equals*

$$\rho_n(t) = \psi(t)^{\otimes n}\otimes\overline{\psi(t)^{\otimes n}} \tag{4.6}$$

where $\psi(t)$ solves Eq. (1.3) with $\alpha = \int_{\mathbb{R}} V(x)\,dx$.

Proof. By Theorem (3.3) we know that ρ_n solves hierarchy (1.21) in the sense of distributions. Due to the form of equations in (1.21), ρ_n must belong to $\mathcal{C}^1(\mathbb{R}, E_n)$, and since the weak-⋆ limit has to be performed in the closed ball centered at the origin of E_n with radius M_1^n, we conclude that ρ_n fulfils the growth condition in the hypothesis of Theorem (4.1) with $K = M_1$. By direct inspection it is easily seen that the density matrices in the r.h.s. of (4.6) solve the same hierarchy and satisfy the growth condition. By Theorem (4.1), the result is proven. □

Remark 4.3. The unique solution of the infinite BBGKY hierarchy can be represented by its Duhamel series. This representation may be used also to show uniqueness in a direct way, as done in Ref. 3, Sec. 5. More specifically, for the solution of (1.21) with initial data ρ_n^I one has

$$\rho_n(t) = U_n(t)\rho_n^I + \sum_{k=1}^{m}(-i)^k \int_{S_k(t)} dT_k\, U_n(t-t_1) \times L_{n,n+1}U_{n+1}(t_1-t_2)\dots L_{n+k-1,n+k}U_{n+k}(t_k)\rho_{n+k}^I + R_{n,m}(t) \tag{4.7}$$

where $dT_k = dt_1\dots dt_k$, $S_k(t)$ is the k-dimensional simplex of size t, namely

$$S_k(t) = \{(s_1,\dots,s_k),\quad 0 \le s_1 \le \cdots \le s_k \le t\} \tag{4.8}$$

and the rest $R_{n,m}(t)$ is given by

$$R_{n,m}(t) := (-i)^{m+1}\int_{S_{m+1}(t)} dT_{m+1}\, U_n(t-t_1)L_{n,n+1}U_{n+1}(t_1-t_2)\dots \dots L_{n+m,n+m+1}\rho_{n+m+1}(t_{n+m+1}) \tag{4.9}$$

Estimates (2.1) and (3.1) imply that the Duhamel formula (4.7) can be extended to a converging series expansion. Indeed, notice that for the generic term of the sum one has

$$\left\| \int_{S_k(t)} dT_k\, U_n(t-t_1)L_{n,n+1}U_{n+1}(t_1-t_2)\dots L_{n+k-1,n+k}U_{n+k}(t_k)\rho_{n+k}^I \right\|_{E_n} \le (8\alpha t)^k \binom{n-1}{k} \|\rho_{n+k}^I\|_{E_n} \le [8\alpha(n+1)t]^k \|\rho_{n+k}^I\|_{E_n} \tag{4.10}$$

Recalling that

$$\|\rho_{n+k}^I\|_{E_n} = \|\psi^I\|_{H^1(\mathbb{R})}^{2n+2k} \tag{4.11}$$

we have that the Duhamel series converges for

$$t < \left[8\alpha(n+1)\|\psi^I\|_{H^1(\mathbb{R})}^2\right]^{-1} \tag{4.12}$$

An analogous computation for the rest $R_{n,m}(t)$, together with estimate (3.1) shows that

$$\|R_{n,m}(t)\|_{E_n} \le [8\alpha(n+1)tM_1]^{m+1}\,\frac{M_1^n}{n+1} \tag{4.13}$$

so the rest is vanishing for

$$t < T := (8\alpha(n+1)M_1)^{-1} \tag{4.14}$$

Remark 4.4. Since estimates (2.1) and (3.1) are uniform in time, the construction by series of the unique solution of (1.21) can be iterated for any time.

From Theorems 3.3 and 4.1, the proof of Theorem 1.1 is complete.

5. APPENDIX: THE INITIAL DATA IN THEOREM 1.1

In this appendix we show that the growth condition (1.25) can be fulfilled by a pure factorized state, i.e. in the case $\rho_N^I = \psi^{\otimes N} \otimes \overline{\psi}^{\otimes N}$. In fact, our proof holds only in the case $\gamma < 1/2$ and we do not know whether for $\gamma \geq 1/2$ there actually are factorized initial data that satisfy (1.25). In the latter case, however, it is possible to mimic the strategy set up by Erdős, Schlein and Yau for the three-dimensional problem[7]: first, approximate the factorized initial data with a non factorized one, say $\tilde{\Psi}_N^I$, that fulfils (1.25); then, apply Theorem (1.1) to the problem (1.1) with initial data $\tilde{\Psi}_N^I$. Finally, remove the approximation on the initial data. This method works also for our problem with $0 < \gamma < 1$. None the less, we prefer to follow another line, close to the one used by Erdős and Yau in the derivation of the Schrödinger-Poisson equation.[9] We have to proceed more carefully, since in our model the range of the potential is shrinking as N grows.

Lemma 5.1. *Let i, j, k, l be distinct elements of $\{1, \dots, N\}$, with $i < j$, $k < l$. Consider two integrable functions $U, W : \mathbb{R} \to \mathbb{R}$ and denote by U_{ij} and W_{kl} the multiplication by $N^{\gamma-1}U(N^\gamma(x_i - x_j))$ and $N^{\gamma-1}W(N^\gamma(x_k - x_l))$. Then, the following inequality holds in the sense of the operators in $L^2(\mathbb{R})$*

$$U_{ij}W_{kl} \leq N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})}S_\lambda^2 S_\mu^2 \tag{5.1}$$

where λ is either i or j, and μ is either k or l.

Proof. Let φ be a smooth, compactly supported function from $\mathbb{R}^N$ to $\mathbb{R}$. Then,

$$\begin{aligned}(\varphi, U_{ij}W_{kl}\varphi) &= \int_{\mathbb{R}^N} |\varphi(X_N)|^2 N^{2\gamma-2} U(N^\gamma(x_i - x_j))W(N^\gamma(x_k - x_l))\, dX_N \\ &= N^{2\gamma-2}\int_{\mathbb{R}} d\zeta_i\, U(N^\gamma\zeta_i)\int_{\mathbb{R}} d\zeta_k\, W(N^\gamma\zeta_k) \\ &\quad \times \int_{\mathbb{R}^{N-2}} d\hat{X}_N^{i,k}\big|\varphi\big(\hat{X}_N^{i,k}, \zeta_i + x_j, \zeta_k + x_l\big)\big|^2\end{aligned} \tag{5.2}$$

where we performed the changes of variables $\zeta_i = x_i - x_j$, $\zeta_k = x_k - x_l$, and introduced the symbol $\hat{X}_N^{i,k}$ denoting the $N-2$-dimensional vector that equals X_N without the components ith and kth. Moreover, with a slight abuse, we denoted $\varphi(\hat{X}_N^{i,k}, \zeta_i + x_j, \zeta_k + x_l)$ the quantity $\varphi(X_N)$ with the argument expressed in the variables $\hat{X}_N^{i,k}, \zeta_i, \zeta_k$. Then, integrating in ζ_i and ζ_k one easily obtains

$$(\varphi, U_{ij}W_{kl}\varphi) \leq N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})} \sup_{\zeta_i,\zeta_k}\int_{\mathbb{R}^{N-2}}\big|\varphi\big(\hat{X}_N^{i,k}, \zeta_i + x_j, \zeta_k + x_l\big)\big|^2 d\hat{X}_N^{i,k}$$

$$\leq N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})}\int_{\mathbb{R}^2} d\zeta_i d\zeta_k$$
$$\times\left|\int_{\mathbb{R}^{N-2}} d\hat{X}_N^{i,k}\partial_{\zeta_i}\partial_{\zeta_k}\left|\varphi\left(\hat{X}_N^{i,k},\zeta_i+x_j,\zeta_k+x_l\right)\right|^2\right| \tag{5.3}$$

where for both variables ζ_i and ζ_k we exploited the estimate

$$\|f\|_{L^\infty(\mathbb{R})}\leq\|f'\|_{L^1(\mathbb{R})} \tag{5.4}$$

that holds for any absolute continuous function f vanishing at infinity. Letting the moduli pass through the sign of integral, after a change of variable we obtain

$$(\varphi,U_{ij}W_{kl}\varphi)\ \leq\ N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})}\int_{\mathbb{R}^N}\left|\partial_{x_i}\partial_{x_k}|\varphi|^2\right|(X_N)\,dX_N \tag{5.5}$$

Applying Leibniz's rule and Cauchy-Schwarz's inequality we get

$$(\varphi,U_{ij}W_{kl}\varphi)\leq 2N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})}$$
$$\times\left(\|\partial_{x_i}\partial_{x_k}\varphi\|_{L^2(\mathbb{R}^N)}\|\varphi\|_{L^2(\mathbb{R}^N)}+\|\partial_{x_i}\varphi\|_{L^2(\mathbb{R}^N)}\|\partial_{x_k}\varphi\|_{L^2(\mathbb{R}^N)}\right)$$
$$\leq N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})}\left(\|\partial_{x_i}\partial_{x_k}\varphi\|^2_{L^2(\mathbb{R}^N)}+\|\varphi\|^2_{L^2(\mathbb{R}^N)}+\|\partial_{x_i}\varphi\|^2_{L^2(\mathbb{R}^N)}\right.$$
$$\left.+\|\partial_{x_k}\varphi\|^2_{L^2(\mathbb{R}^N)}\right)=N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})}\left(\varphi,S_i^2,S_k^2\varphi\right) \tag{5.6}$$

Remarking that indices i and j, as well as k and l, are exchangeable, we complete the proof. □

Lemma 5.2. *Let i,j,k be distinct elements of $\{1,\ldots,N\}$, with $i<j$, $i<l$. Consider two integrable functions $U,W:\mathbb{R}\to\mathbb{R}$ and denote by U_{ij} and W_{il} the multiplication by $N^{\gamma-1}U(N^\gamma(x_i-x_j))$ and $N^{\gamma-1}W(N^\gamma(x_i-x_l))$. Then, the following inequality holds in the sense of the operators in $L^2(\mathbb{R})$*

$$U_{ij}W_{il}\leq N^{-2}\|U\|_{L^1(\mathbb{R})}\|W\|_{L^1(\mathbb{R})}S_j^2S_l^2 \tag{5.7}$$

Proof. The proof of the preceding lemma can be replicated replacing i by j. □

Lemma 5.3. *Let i,j belong to $\{1,\ldots,N\}$, with $i<j$. Consider two integrable functions $U,W:\mathbb{R}\to\mathbb{R}$ and denote by U_{ij} and W_{ij} the multiplication by $N^{\gamma-1}U(N^\gamma(x_i-x_j))$ and $N^{\gamma-1}W(N^\gamma(x_i-x_j))$. Then, the following inequality holds in the sense of the operators in $L^2(\mathbb{R})$*

$$U_{ij}W_{ij}\leq N^{2\gamma-2}\|U\|_{L^\infty(\mathbb{R})}\|W\|_{L^\infty(\mathbb{R})} \tag{5.8}$$

Proof. The proof is trivial. □

Now we introduce the main technical lemmas.

Lemma 5.4. *Let γ be less than $1/2$, denote by V_{ij} the multiplication by $N^{\gamma-1}V(N^\gamma(x_i - x_j))$, and by Δ the laplacian with respect to the variables $\{x_1, \ldots, x_N\}$. Then, for any $p \in \mathbb{N}$ the following inequality holds in the sense of the operators in $L^2(\mathbb{R}^N)$*

$$\sum_{\substack{i<j,k<l \\ \#(\{i,j,k,l\})=4}} V_{ij}(-\Delta)^p V_{kl} \le \|V\|^2_{L^1(\mathbb{R})}(-\Delta)^p(N-\Delta)^2 + 7^p \|V\|^2_{W^{p,1}(\mathbb{R})} \times \left(N^{-1} + N^{2\gamma-1}\right)(N-\Delta)^{p+2} \tag{5.9}$$

Proof. From now on we shall make use of the symbol $D_i = -i\partial_{x_i}$, and denote by $A^\star$ the adjoint in L^2 of the operator A. Furthermore, $A +$ h.c. will denote the operator $A + A^\star$.

Applying Leibniz's rule,

$$\begin{aligned}
V_{ij}(-\Delta)^p V_{kl} = & \sum_{\alpha_1,\ldots,\alpha_p=1}^{N} \left(D_{\alpha_1}\ldots D_{\alpha_p} V_{ij}\right)^\star D_{\alpha_1}\ldots D_{\alpha_p} V_{kl} \\
= & \sum_{q=0}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} \sum_{q_k=0}^{q-q_i-q_j} \binom{q-q_i-q_j}{q_k} \\
& \times \sum_{\substack{\beta_1,\ldots,\beta_{p-q} \\ \neq i,j,k,l}} D_{\beta_1}\ldots D_{\beta_{p-q}} \left(D_i^{q_i} D_j^{q_j} D_k^{q_k} D_l^{q_l} V_{ij}\right)^\star \\
& \times \left(D_i^{q_i} D_j^{q_j} D_k^{q_k} D_l^{q_l} V_{kl}\right) D_{\beta_1}\ldots D_{\beta_{p-q}} \\
= & \sum_{\substack{\beta_1,\ldots,\beta_p \\ \neq i,j,k,l}} D_{\beta_1}\ldots D_{\beta_p} V_{ij} V_{kl} D_{\beta_1}\ldots D_{\beta_p} \\
& + \sum_{q=1}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} \sum_{q_k=0}^{q-q_i-q_j} \binom{q-q_i-q_j}{q_k} \\
& \times \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} \sum_{m_k=0}^{q_k} \binom{q_k}{m_k} \\
& \times \sum_{m_l=0}^{q_l} \binom{q_l}{m_l} i^{m_i-m_j-m_k+m_l} N^{\gamma(m_i+m_j+m_k+m_l)} \\
& \times \sum_{\substack{\beta_1,\ldots,\beta_{p-q} \\ \neq i,j,k,l}} D_{\beta_1}\ldots D_{\beta_{p-q}} D_k^{q_k} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)} \\
& \times D_k^{q_k-m_k} D_l^{q_l-m_l} D_i^{q_i} D_j^{q_j} D_{\beta_1}\ldots D_{\beta_{p-q}} = (I) + (II)
\end{aligned} \tag{5.10}$$

where we defined $q_l = q - q_i - q_j - q_k$ and

$$V_{uv}^{(w)} = N^{\gamma-1} V^{(w)}(N^\gamma(x_u - x_v)) \tag{5.11}$$

An estimate for the operator (I) is obtained applying lemma (5.1) with $U = W = V$. Indeed,

$$(I) \leq \; N^{-2}\|V\|^2_{L^1(\mathbb{R})} S_i^2 S_k^2 (-\Delta)^p \tag{5.12}$$

Summing all terms like (5.12) over all distinct pairs $i < j$ and $k < l$, we obtain the first term in the r.h.s. of (5.9).

Let us discuss the term (II) in (5.10).
Using the notation

$$\begin{aligned} A =\; & i^{m_j - m_i} N^{\gamma(m_i+m_j)} \mathrm{sgn}\left(V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)}\right) \left|V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)}\right|^{1/2} \\ & \times D_k^{q_k} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} \\ B =\; & i^{m_l-m_k} N^{\gamma(m_k+m_l)} \left|V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)}\right|^{1/2} D_k^{q_k-m_k} D_l^{q_l-m_l} D_i^{q_i} D_j^{q_j} \end{aligned} \tag{5.13}$$

we notice that

$$\begin{aligned} & i^{m_i-m_j-m_k+m_l} N^{\gamma(m_i+m_j+m_k+m_l)} D_k^{q_k} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)} D_k^{q_k-m_k} \\ & \times D_l^{q_l-m_l} D_i^{q_i} D_j^{q_j} + \text{h.c.} \\ & = A^\star B + B^\star A \;\leq\; A^\star A + B^\star B = \; N^{2\gamma(m_i+m_j)} D_k^{q_k} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} \\ & \times \left|V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)}\right| D_k^{q_k} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} \\ & + N^{2\gamma(m_k+m_l)} D_k^{q_k-m_k} D_l^{q_l-m_l} D_i^{q_i} D_j^{q_j} \left|V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)}\right| D_k^{q_k-m_k} D_l^{q_l-m_l} D_i^{q_i} D_j^{q_j} \end{aligned} \tag{5.14}$$

Exploiting Lemma 5.1, with $U = \left|V^{(m_i+m_j)}\right|$ and $W = \left|V^{(m_k+m_l)}\right|$, we finally obtain

$$\begin{aligned} & i^{m_i-m_j-m_k+m_l} N^{\gamma(m_i+m_j+m_k+m_l)} D_k^{q_k} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} V_{ij}^{(m_i+m_j)} V_{kl}^{(m_k+m_l)} D_k^{q_k-m_k} \\ & \times D_l^{q_l-m_l} D_i^{q_i} D_j^{q_j} + \text{h.c.} \leq N^{2\gamma(m_i+m_j)-2} \|V^{(m_i+m_j)}\|_{L^1(\mathbb{R})} \|V^{(m_k+m_l)}\|_{L^1(\mathbb{R})} \\ & \times S_\lambda^2 S_\mu^2 D_k^{2q_k} D_l^{2q_l} D_i^{2(q_i-m_i)} D_j^{2(q_j-m_j)} + N^{2\gamma(m_k+m_l)-2} \|V^{(m_i+m_j)}\|_{L^1(\mathbb{R})} \\ & \times \|V^{(m_k+m_l)}\|_{L^1(\mathbb{R})} S_\lambda^2 S_\mu^2 D_k^{2(q_k-m_k)} D_l^{2(q_l-m_l)} D_i^{2q_i} D_j^{2q_j} \end{aligned} \tag{5.15}$$

Then, by (5.10) one finally obtains

$$\begin{aligned}
&(II) + \text{h.c.} \\
&\leq \sum_{q=1}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} \sum_{q_k=0}^{q-q_i-q_j} \binom{q-q_i-q_j}{q_k} \\
&\quad \times \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} \sum_{m_k=0}^{q_k} \binom{q_k}{m_k} \sum_{m_l=0}^{q_l} \binom{q_l}{m_l} (-\Delta)^{p-q} \\
&\quad \times \Big\{ N^{2\gamma(m_i+m_j)-2} \| V^{(m_i+m_j)} \|_{L^1(\mathbb{R})} \| V^{(m_k+m_l)} \|_{L^1(\mathbb{R})} S_\lambda^2 S_\mu^2 D_k^{2q_k} D_l^{2q_l} D_i^{2(q_i-m_i)} \\
&\quad \times D_j^{2(q_j-m_j)} + N^{2\gamma(m_k+m_l)-2} \| V^{(m_i+m_j)} \|_{L^1(\mathbb{R})} \| V^{(m_k+m_l)} \|_{L^1(\mathbb{R})} S_\lambda^2 S_\mu^2 D_k^{2(q_k-m_k)} \\
&\quad \times D_l^{2(q_l-m_l)} D_i^{2q_i} D_j^{2q_j} \Big\} \qquad (5.16) \\
&\leq \| V \|_{W^{p,1}(\mathbb{R})}^2 \sum_{q=1}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} \sum_{q_k=0}^{q-q_i-q_j} \binom{q-q_i-q_j}{q_k} \\
&\quad \times \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} \sum_{m_k=0}^{q_k} \binom{q_k}{m_k} \sum_{m_l=0}^{q_l} \binom{q_l}{m_l} (-\Delta)^{p-q} \\
&\quad \times \Big\{ N^{2\gamma(m_i+m_j)-2} S_\lambda^2 S_\mu^2 D_k^{2q_k} D_l^{2q_l} D_i^{2(q_i-m_i)} D_j^{2(q_j-m_j)} \\
&\quad + N^{2\gamma(m_k+m_l)-2} S_\lambda^2 S_\mu^2 D_k^{2(q_k-m_k)} D_l^{2(q_l-m_l)} D_i^{2q_i} D_j^{2q_j} \Big\} \\
&= (III) + (IV)
\end{aligned}$$

Let us consider (III), namely the sums applied to the first term between graph parentheses. First observe that the sums in m_k and m_l are trivial and result in the factor $2^{q-q_i-q_j}$. Moreover, it is convenient to split (III) in two further terms, one referred to the case $m_i = m_j = 0$ and the other collecting the rest. In fact we obtain

$$\begin{aligned}
(III) = N^{-2} \| V \|_{W^{p,1}(\mathbb{R})}^2 \sum_{q=1}^{p} \binom{p}{q} (-\Delta)^{p-q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} 2^{q-q_i-q_j} \\
\times \sum_{q_k=0}^{q-q_i-q_j} \binom{q-q_i-q_j}{q_k} \times S_\lambda^2 S_\mu^2 D_k^{2q_k} D_l^{2q_l} D_i^{2q_i} D_j^{2q_j}
\end{aligned}$$

$$+ N^{-2}\|V\|^2_{W^{p,1}(\mathbb{R})}\sum_{q=1}^{p}\binom{p}{q}(-\Delta)^{p-q}\sum_{q_i=0}^{q}\binom{q}{q_i}\sum_{q_j=0}^{q-q_i}\binom{q-q_i}{q_j}2^{q-q_i-q_j}$$

$$\times \sum_{q_k=0}^{q-q_i-q_j}\binom{q-q_i-q_j}{q_k}\sum_{m_i=0}^{q_i}\binom{q_i}{m_i}$$

$$\times \sum_{\substack{m_j=0\\ m_j+m_i>0}}^{q_j}\binom{q_j}{m_j}N^{2\gamma(m_i+m_j)}S_\lambda^2 S_\mu^2 D_k^{2q_k}D_l^{2q_l}D_i^{2(q_i-m_i)}D_j^{2(q_j-m_j)}$$

$$= (III-A)_{ijkl} + (III-B)_{ijkl} \tag{5.17}$$

Concerning $(III-A)_{ijkl}$, we notice that at least one among the numbers q_i, q_j, q_k, q_l has to be non zero. Let us suppose that it is q_i, then choose $\lambda = j$, and μ arbitrarily either k or l. Notice that

$$N^{-2}\sum_{i<j,k<l} S_j^2 S_\mu^2 D_k^{2q_k}D_l^{2q_l}D_i^{2q_i}D_j^{2q_j} \;\le\; N^{-1}(-\Delta)^q(N-\Delta)^2 \tag{5.18}$$

due to the fact that in the sum in the r.h.s. there are at least three distinct indices: i, j, and μ. Notice that the sum concerns the derivative indices only, not the powers. Therefore, performing the sum over all pairs we finally obtain

$$\sum_{i<j,k<l}(III-A)_{ijkl} \le N^{-1}\|V\|^2_{W^{p,1}(\mathbb{R})}(N-\Delta)^2(-\Delta)^p$$

$$\times\sum_{q=1}^{p}\binom{p}{q}\sum_{q_i=0}^{q}\binom{q}{q_i}\sum_{q_j=0}^{q-q_i}\binom{q-q_i}{q_j}4^{q-q_i-q_j}$$

$$\le 7^p N^{-1}\|V\|^2_{W^{p,1}(\mathbb{R})}(N-\Delta)^{p+2} \tag{5.19}$$

Concerning the terms of the type $(III-B)_{ijkl}$, we first observe that

$$N^{-2}\sum_{i<j,k<l} S_\lambda^2 S_\mu^2 D_k^{2q_k}D_l^{2q_l}D_i^{2(q_i-m_i)}D_j^{2(q_j-m_j)} \;\le\; (-\Delta)^{q-m_i-m_j}(N-\Delta)^2 \tag{5.20}$$

Notice that such estimate is worse than (5.18): this is due to the fact that the four indices q_k, q_l, q_i-m_i and q_j-m_j could vanish simultaneously, regardless of the choice of λ and μ. Moreover, since $m_i+m_j>0$, we have

$$N^{2\gamma(m_i+m_j)} \le N^{2\gamma-1}N^{m_i+m_j} \tag{5.21}$$

Therefore

$$\sum_{i<j,k<l}(III-B)_{ijkl} \le N^{2\gamma-1}\|V\|^2_{W^{p,1}(\mathbb{R})}(N-\Delta)^2\sum_{q=1}^{p}\binom{p}{q}\sum_{q_i=0}^{q}\binom{q}{q_i}$$

$$\times \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} 4^{q-q_i-q_j} \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} N^{m_i+m_j} (-\Delta)^{p-m_i-m_j}$$

$$\leq 7^p N^{2\gamma-1} \|V\|^2_{W^{p,1}(\mathbb{R})} (N-\Delta)^{p+2} \tag{5.22}$$

From (5.22) and (5.19) we can conclude

$$(III) \leq 7^p \|V\|^2_{W^{p,1}(\mathbb{R})} \left(N^{2\gamma-1} + N^{-1}\right) (N-\Delta)^{p+2} \tag{5.23}$$

Remarking that an identical estimate holds for (IV), and recalling estimate (5.12) we obtain

$$\sum_{\substack{i<j,k<l \\ \#(\{i,j,k,l\})=4}} \left(V_{ij}(-\Delta)^p V_{kl} + V_{kl}(-\Delta)^p V_{ij}\right) \leq 2\|V\|^2_{L^1(\mathbb{R})} (-\Delta)^p (N-\Delta)^2$$
$$+ 2 \cdot 7^p \|V\|^2_{W^{p,1}(\mathbb{R})} \left(N^{-1} + N^{2\gamma-1}\right) (N-\Delta)^{p+2} \tag{5.24}$$

Observing that

$$\sum_{\substack{i<j,k<l \\ \#(\{i,j,k,l\})=4}} V_{ij}(-\Delta)^p V_{kl} = \frac{1}{2} \sum_{\substack{i<j,k<l \\ \#(\{i,j,k,l\})=4}} \left(V_{ij}(-\Delta)^p V_{kl} + V_{kl}(-\Delta)^p V_{ij}\right) \tag{5.25}$$

we obtain inequality (5.9) and the proof is complete. □

Lemma 5.5. *Let γ be less than $1/2$, denote by V_{uv} the multiplication by $N^{\gamma-1}V(N^\gamma(x_u - x_v))$, and by Δ the laplacian with respect to the variables $\{x_1, \dots, x_N\}$. Then, for any $p \in \mathbb{N}$ the following inequality holds in the sense of the operators in $L^2(\mathbb{R}^N)$*

$$\sum_{\substack{i<j,i<l \\ \#(\{i,j,l\})=3}} V_{ij}(-\Delta)^p V_{il} \leq 5^p N^{-1} \|V\|^2_{W^{p,1}(\mathbb{R})} (N-\Delta)^{p+2} \tag{5.26}$$

Proof. We follow the line of lemma 5.4.

$$V_{ij}(-\Delta)^p V_{il} \leq \sum_{q=0}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j}$$

$$\times \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} \sum_{m_i'=0}^{q_i} \binom{q_i}{m_i'} \sum_{m_l=0}^{q_l} \binom{q_l}{m_l} i^{m_i-m_j-m_i'+m_l} N^{\gamma(m_i+m_j+m_i'+m_l)}$$

$$\times \sum_{\substack{\beta_1,\ldots,\beta_{p-q} \\ \neq i,j,l}} D_{\beta_1}\ldots D_{\beta_{p-q}} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)} D_i^{q_i-m_i'} D_l^{q_l-m_l}$$
$$\times D_j^{q_j} D_{\beta_1}\ldots D_{\beta_{p-q}} \tag{5.27}$$

where we defined $q_l = q - q_i - q_j$ and

$$V_{uv}^{(w)} = N^{\gamma-1} V^{(w)}(N^\gamma(x_u - x_v)) \tag{5.28}$$

Using the notation

$$\begin{aligned} A =\ & i^{m_j-m_i} N^{\gamma(m_i+m_j)} \mathrm{sgn}\left(V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)}\right) \left|V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)}\right|^{1/2} \\ & \times D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} \\ B =\ & i^{m_l-m_i'} N^{\gamma(m_i'+m_l)} \left|V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)}\right|^{1/2} D_l^{q_l-m_l} D_i^{q_i-m_i'} D_j^{q_j} \end{aligned} \tag{5.29}$$

we notice that

$$\begin{aligned} & i^{m_i-m_j+m_l-m_i'} N^{\gamma(m_i+m_i'+m_j+m_l)} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)} D_l^{q_l-m_l} D_i^{q_i-m_i'} \\ & \times D_j^{q_j} + \text{h.c.} \le\ A^\star A + B^\star B\ =\ N^{2\gamma(m_i+m_j)} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} \\ & \times \left|V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)}\right| D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} + N^{2\gamma(m_i'+m_l)} D_l^{q_l-m_l} D_i^{q_i-m_i'} D_j^{q_j} \\ & \times \left|V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)}\right| D_l^{q_l-m_l} D_i^{q_i-m_i'} D_j^{q_j} \end{aligned} \tag{5.30}$$

Exploiting Lemma 5.2, with $U = \left|V^{(m_i+m_j)}\right|$ and $W = \left|V^{(m_i'+m_l)}\right|$, we obtain

$$\begin{aligned} & i^{m_i-m_j+m_l-m_i'} N^{\gamma(m_i+m_i'+m_j+m_l)} D_l^{q_l} D_i^{q_i-m_i} D_j^{q_j-m_j} V_{ij}^{(m_i+m_j)} V_{il}^{(m_i'+m_l)} D_l^{q_l-m_l} D_i^{q_i-m_i'} \\ & \times D_j^{q_j} + \text{h.c.} \le N^{2\gamma(m_i+m_j)-2} \| V^{(m_i+m_j)}\|_{L^1(\mathbb{R})} \| V^{(m_i'+m_l)}\|_{L^1(\mathbb{R})} \\ & \times S_j^2 S_l^2 D_l^{2q_l} D_i^{2(q_i-m_i)} D_j^{2(q_j-m_j)} + N^{2\gamma(m_i'+m_l)-2} \| V^{(m_i+m_j)}\|_{L^1(\mathbb{R})} \| V^{(m_i'+m_l)}\|_{L^1(\mathbb{R})} \\ & \times S_j^2 S_l^2 D_l^{2(q_l-m_l)} D_i^{2(q_i-m_i')} D_j^{2q_j} \end{aligned} \tag{5.31}$$

Then, by (5.27) one finally gets

$$\begin{aligned} & V_{ij}(-\Delta)^p V_{il} + \text{h.c.} \le \|V\|^2_{W^{p,1}(\mathbb{R})} \\ & \times \sum_{q=0}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} \sum_{m_i'=0}^{q_i} \binom{q_i}{m_i'} \end{aligned}$$

$$\times \sum_{m_l=0}^{q_l} \binom{q_l}{m_l} (-\Delta)^{p-q} \left\{ N^{2\gamma(m_i+m_j)-2} S_j^2 S_l^2 D_l^{2q_l} D_i^{2(q_i-m_i)} D_j^{2(q_j-m_j)} \right.$$

$$\left. + N^{2\gamma(m_i'+m_l)-2} S_j^2 S_l^2 D_l^{2(q_l-m_l)} D_i^{2(q_i-m_i')} D_j^{2q_j} \right\} \tag{5.32}$$

Let us remark that

$$N^{-2} \sum_{i,j,l=1}^{N} S_j^2 S_l^2 D_l^{2q_l} D_i^{2(q_i-m_i)} D_j^{2(q_j-m_j)} \le N^{-1}(N-\Delta)^2(-\Delta)^{q-m_i-m_j} \tag{5.33}$$

Then an easy computation based on Newton binomial gives

$$\sum_{\substack{i<j,i<l \\ \#(\{i,j,l\})=3}} (V_{ij}(-\Delta)^p V_{il} + \text{h.c.})$$

$$\le 2N^{-1} \|V\|^2_{W^{p,1}(\mathbb{R})} (N-\Delta)^2 \sum_{q=0}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i}$$

$$\times \sum_{q_j=0}^{q-q_i} \binom{q-q_i}{q_j} 2^{q-q_j} (-\Delta)^{q-q_i-q_j} (1-\Delta)^{q_i+q_j}$$

$$\le 2 \cdot 5^p N^{-1} \|V\|^2_{W^{p,1}(\mathbb{R})} (N-\Delta)^{p+2} \tag{5.34}$$

so the proof is complete. □

Lemma 5.6. *Let γ be less than $1/2$, denote by V_{ij} the multiplication by $N^{\gamma-1}V(N^\gamma(x_i - x_j))$, and by Δ the laplacian with respect to the variables $\{x_1, \ldots, x_N\}$. Then, for any $p \in \mathbb{N}$ the following inequality holds in the sense of the operators in $L^2(\mathbb{R}^N)$*

$$\sum_{i<j} V_{ij}(-\Delta)^p V_{ij} \le 5^p N^{2\gamma-2} \|V\|^2_{W^{p,\infty}(\mathbb{R})} (N-\Delta)^{p+2} \tag{5.35}$$

Proof. We follow the line of Lemma 5.4.

$$V_{ij}(-\Delta)^p V_{ij} \le \sum_{q=0}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} \sum_{m_i'=0}^{q_i} \binom{q_i}{m_i'}$$

$$\times \sum_{m_j'=0}^{q_j} \binom{q_j}{m_j'} i^{m_i-m_j+m_j'-m_i'} N^{\gamma(m_i+m_j+m_i'+m_j')}$$

$$\times \sum_{\substack{\beta_1,\dots,\beta_{p-q} \\ \neq i,j}} D_{\beta_1} \dots D_{\beta_{p-q}} D_i^{q_i - m_i} D_j^{q_j - m_j} V_{ij}^{(m_i + m_j)} V_{ij}^{(m'_i + m'_j)} D_i^{q_i - m'_i}$$

$$\times D_j^{q_j - m'_j} D_{\beta_1} \dots D_{\beta_{p-q}} \tag{5.36}$$

where we defined $q_j = q - q_i$ and

$$V_{ij}^{(w)} = N^{\gamma - 1} V^{(w)}(N^\gamma (x_i - x_j)) \tag{5.37}$$

Using the notation

$$A = i^{m_j - m_i} N^{\gamma(m_i + m_j)} \mathrm{sgn}\left(V_{ij}^{(m_i+m_j)} V_{ij}^{(m'_i+m'_j)} \right) \left| V_{ij}^{(m_i+m_j)} V_{ij}^{(m'_i+m'_j)} \right|^{1/2}$$
$$\times D_i^{q_i - m_i} D_j^{q_j - m_j}$$
$$B = i^{m'_j - m'_i} N^{\gamma(m'_i + m'_j)} \left| V_{ij}^{(m_i+m_j)} V_{ij}^{(m'_i+m'_j)} \right|^{1/2} D_i^{q_i - m'_i} D_j^{q_j - m'_j} \tag{5.38}$$

we notice that

$$i^{m_i - m_j + m'_j - m'_i} N^{\gamma(m_i + m_j + m'_i + m'_j)} D_i^{q_i - m_i} D_j^{q_j - m_j} V_{ij}^{(m_i+m_j)} V_{ij}^{(m'_i+m'_j)} D_i^{q_i - m'_i} D_j^{q_j - m'_j}$$
$$+ \text{h.c.} \ \leq \ A^\star A + B^\star B$$
$$= N^{2\gamma(m_i+m_j)} D_i^{q_i - m_i} D_j^{q_j - m_j} \left| V_{ij}^{(m_i+m_j)} V_{ij}^{(m'_i+m'_j)} \right| D_i^{q_i - m_i} D_j^{q_j - m_j}$$
$$+ N^{2\gamma(m'_i+m'_j)} D_i^{q_i - m'_i} D_j^{q_j - m'_j} \left| V_{ij}^{(m_i+m_j)} V_{ij}^{(m'_i+m'_j)} \right| D_i^{q_i - m'_i} D_j^{q_j - m'_j} \tag{5.39}$$

Exploiting Lemma 5.3, with $U = \left| V^{(m_i+m_j)} \right|$ and $W = \left| V^{(m'_i+m'_j)} \right|$, we obtain

$$i^{m_i - m_j + m'_j - m'_i} N^{\gamma(m_i + m'_i + m_j + m'_j)} D_i^{q_i - m_i} D_j^{q_j - m_j} V_{ij}^{(m_i+m_j)} V_{ij}^{(m'_i+m'_j)} D_i^{q_i - m'_i}$$
$$\times D_j^{q_j - m'_j} + \text{h.c.} \leq N^{2\gamma(m_i+m_j+1)-2} \| V^{(m_i+m_j)} \|_{L^\infty(\mathbb{R})} \| V^{(m'_i+m'_j)} \|_{L^\infty(\mathbb{R})}$$
$$\times D_i^{2(q_i - m_i)} D_j^{2(q_j - m_j)} + N^{2\gamma(m'_i+m'_j+1)-2} \| V^{(m_i+m_j)} \|_{L^\infty(\mathbb{R})} \| V^{(m'_i+m'_j)} \|_{L^\infty(\mathbb{R})}$$
$$\times D_i^{2(q_i - m'_i)} D_j^{2(q_j - m'_j)} \tag{5.40}$$

Then, by (5.36) one finally obtains

$$2\, V_{ij} (-\Delta)^p V_{ij} \leq \| V \|^2_{W^{p,\infty}(\mathbb{R})}$$
$$\times \sum_{q=0}^{p} \binom{p}{q} \sum_{q_i=0}^{q} \binom{q}{q_i} \sum_{m_i=0}^{q_i} \binom{q_i}{m_i} \sum_{m_j=0}^{q_j} \binom{q_j}{m_j} \sum_{m'_i=0}^{q_i} \binom{q_i}{m'_i}$$

$$\times \sum_{m'_j=0}^{q_j} \binom{q_j}{m'_j} (-\Delta)^{p-q} \left\{ N^{2\gamma(m_i+m_j+1)-2} D_i^{2(q_i-m_i)} D_j^{2(q_j-m_j)} \right.$$

$$\left. + N^{2\gamma(m'_i+m'_j+1)-2} D_i^{2(q_i-m'_i)} D_j^{2(q_j-m'_j)} \right\} \tag{5.41}$$

We remark that

$$N^{-2} \sum_{i,j=1}^{N} D_i^{2(q_i-m_i)} D_j^{2(q_j-m_j)} \leq (-\Delta)^{q_i+q_j-m_i-m_j} \tag{5.42}$$

Then, performing the sums in (5.41) according to Newton binomials, we obtain

$$\sum_{i<j} V_{ij}(-\Delta)^p V_{ij} \leq \|V\|^2_{W^{p,\infty}(\mathbb{R})} N^{2\gamma} \left(4N^{2\gamma} - 5\Delta\right)^p \tag{5.43}$$

so estimate (5.35) easily follows and the proof is complete. □ We are now ready to prove the following

Proposition 5.7. *The following inequality holds in the sense of the operators in* $L^2(\mathbb{R}^N)$

$$\mathbb{V}(-\Delta)^p\mathbb{V} \leq \|V\|^2_{L^1(\mathbb{R})}(-\Delta)^p(N-\Delta)^2 + \left[(7^p + 4\cdot 5^p)\|V\|^2_{W^{p,1}(\mathbb{R})}N^{-1}\right.$$
$$\left. + 7^p\|V\|^2_{W^{p,1}(\mathbb{R})}N^{2\gamma-1} + 5^p\|V\|^2_{W^{\infty,1}(\mathbb{R})}N^{2\gamma-2}\right](N-\Delta)^{p+2} \tag{5.44}$$

where

$$\mathbb{V} = \sum_{i<j} V_{ij}, \quad V_{ij} = N^{\gamma-1}V(N^\gamma(x_i - x_j)) \tag{5.45}$$

and $\gamma < 1/2$.

Proof. Denoting by # the cardinality of the set $\{i, j, k, l\}$ one has

$$\sum_{i<j,k<l} V_{ij}(-\Delta)^p V_{kl} = \sum_{\substack{i<j,k<l\\ \#=4}} V_{ij}(-\Delta)^p V_{kl} + \sum_{\substack{i<j,k<l\\ \#=3}} V_{ij}(-\Delta)^p V_{kl} + \sum_{\substack{i<j,k<l\\ \#=2}} V_{ij}(-\Delta)^p V_{kl}$$

$$= \sum_{\substack{i<j,k<l\\ \#=4}} V_{ij}(-\Delta)^p V_{kl} + \sum_{i<j<l} V_{ij}(-\Delta)^p V_{il} + \sum_{i<j<l} V_{ij}(-\Delta)^p V_{jl}$$

$$+ \sum_{i<j<l} V_{jl}(-\Delta)^p V_{ij} + \sum_{i<l<j} V_{ij}(-\Delta)^p V_{lj} + \sum_{i<j} V_{ij}(-\Delta)^p V_{ij} \tag{5.46}$$

Since the terms with $\# = 3$ can be estimated as in (5.26), the proposition is proven. $\square$

Now we introduce the approximation theorem for the many-body dynamics.

Theorem 5.8. *For any $n \in \mathbb{N}$ there exists $N^\star(n)$ such that for $N > N^\star(n)$*

$$H_N^n \leq C^n (N - \Delta)^n \tag{5.47}$$

with C independent of n.

Proof. The proof is done by a two-step induction. For $n = 0$ the proposition is trivial with $C \geq 1$. For $n = 1$ we have

$$\begin{aligned}(\varphi, H_N \varphi) \;&\leq\; (\varphi, -\Delta\varphi) + N^{-1} \|V\|_{L^1(\mathbb{R})} \sum_{i \leq j} \int_{\mathbb{R}^{N-1}} d\hat{X}_N^i \sup_{x_i \in \mathbb{R}} |\varphi(X_N)|^2 \\ &\leq (\varphi, -\Delta\varphi) + N^{-1} \|V\|_{L^1(\mathbb{R})} \sum_{i \leq j} \big(\varphi, \big(\mathbb{I} - \partial_{x_i}^2\big)\varphi\big) \\ &\leq \big(\|V\|_{L^1(\mathbb{R}^3)} + 1\big)(\varphi, (N - \Delta)\varphi)\end{aligned} \tag{5.48}$$

Let us suppose that for some n there exists $N^\star(n)$ such that $H^n \leq C^n (N - \Delta)^n$ for $N > N^\star(n)$. Then

$$H_N^{n+2} \;\leq\; C^n H_N (N - \Delta)^n H_N \;\leq\; 2C^n (N - \Delta)^{n+2} + 2C^n \mathbb{V}(N - \Delta)^n \mathbb{V} \tag{5.49}$$

Let us focus on the last term. Using the Newton expansion we find

$$\mathbb{V}(N - \Delta)^n \mathbb{V} \;=\; \sum_{p=0}^{n} \binom{n}{p} N^{n-p} \mathbb{V}(-\Delta)^p \mathbb{V} \tag{5.50}$$

and by inequality (5.44) we have

$$\begin{aligned}\mathbb{V}(N - \Delta)^n \mathbb{V} \;&\leq\; \|V\|_{L^1(\mathbb{R})}^2 (N - \Delta)^2 \sum_{p=0}^{n} \binom{n}{p} N^{n-p} (-\Delta)^p \\ &+ c(N, n) \sum_{p=0}^{n} \binom{n}{p} (N - \Delta)^{p+2}\end{aligned} \tag{5.51}$$

where we defined the quantity

$$\begin{aligned}c(N, n) := (7^n + 4 \cdot 5^n) \|V\|_{W^{n,1}(\mathbb{R})}^2 N^{-1} + 7^n \|V\|_{W^{n,1}(\mathbb{R})}^2 \\ \times N^{2\gamma-1} + 5^n \|V\|_{W^{\infty,1}(\mathbb{R})} N^{2\gamma-2}\end{aligned} \tag{5.52}$$

Since $c(N, n)$ vanishes as $N \to \infty$, there exists $N^\star(n+2)$ such that

$$H_N^{n+2} \leq 2C^n(N-\Delta)^{n+2} + 2C^n \| V \|_{L^1(\mathbb{R})}^2 (N-\Delta)^{n+2} \tag{5.53}$$

for any $N > N^\star(n+2)$. Then, for $C \geq \sqrt{2}(1 + \| V \|_{L^1(\mathbb{R})})$ the theorem is proven. □

It is now clear how to construct the desired initial data: just take a factorized state that realizes the growth condition with the free evolution. For instance, $\Psi_N^I = \psi^{\otimes N}$ with $\hat{\psi}$ having compact support.

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REFERENCES

1. R. Adami, C. Bardos, F. Golse and A. Teta, Towards a rigorous derivation of the cubic NLSE in dimension one. *Asymptot. Anal.* **40**(2):93–108 (2004).
2. C. Bardos, L. Erdős, F. Golse, N. Mauser and H.-T.Yau, Derivation of the Schrödinger-Poisson equation from the quantum N-body problem. C. R. *Math. Acad. Sci.* Paris **334**(6):515–520 (2002).
3. C. Bardos, F. Golse and N. Mauser, Weak coupling limit of the N particles Schrödinger equation. *Methods Appl. Anal.* **7**(2):275–293 (2000).
4. A. Elgart, L. Erdős, B. Schlein and H.-T. Yau, Gross-Pitaevskii equation as the mean field limit of weakly coupled Bosons. *Math. Phys.* preprint archive, mp-arc 04-333 (2004).
5. L. Erdős, private communication.
6. L. Erdős, B. Schlein and H.-T. Yau, Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Math. Phys.* preprint archive, mp-arc 04-319 (2004).
7. L. Erdős, B. Schlein and H.-T. Yau, Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems, mp-arc 05-280.
8. L. Erdős, B. Schlein and H.-T. Yau, Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate, mp-arc 06-176.
9. L. Erdős and H.-T. Yau, Derivation of the nonlinear Schrödinger equation from a many body Coulomb system. *Adv. Theor. Math. Phys.* **5**(6):1169–1205 (2001).
10. J. Ginibre and G. Velo, The classical field limit of scattering theory for non relativistic many-boson systems. I and II. *Comm. Math. Phys.* **66**:37–76 (1979), and **68**:45–68 (1979).
11. K. Hepp, The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.* **35**:265–277 (1974).
12. E.H. Lieb, R. Seiringer and J. Yngvason, The Quantum-mechanical Many Body Problem: The Bose Gas. http://arxiv.org/abs/math-ph/?math-ph%2F0405004.
13. E. H. Lieb, R. Seiringer and J. Yngvason, One-dimensional behavior of dilute, trapped Bose gases. *Comm. Math. Phys.* **244**(2):347–393 (2004).
14. H. Spohn, Kinetic equations from hamiltonian dynamics. *Rev. Mod. Phys.* **52**(3):600–640 (1980).